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Logic Group
Preprint Series
No. 74
February 1992

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A tractable algorithm for the wellfounded model

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January 16, 1992

Abstract

In the area of general logic programming (negated atoms allowed in the bodies of rules) and reason maintenance systems, the wellfounded model (first defined by Van Gelder, Ross and Schlipf in 1988) is generally considered to be *the* declarative semantics of the program. In this paper we present the concise mathematical development of a quadratic-time algorithm for the wellfounded model of propositional general logic programs. This algorithm has essentially been developed by Witteveen in 1990, based on ideas and material by Goodwin (1982).

1. Introduction

For logic programs without negation the declarative semantics is clear: only those propositions which have to be true are true, anything else is false. This is the minimal model semantics. Since these programs have only limited expressive power, extending the language of logic programs is useful. Logic programs where negated atoms in the bodies of rules are allowed, are called *general logic programs* (see e.g. [L87]). The minimal model semantics is not acceptable here, since the truth of a proposition may depend upon the falsity of others. The principle of negation as failure is used to obtain the falsity of atoms.

Several semantics have been proposed for general logic programs (see [PP90], [J91]), culminating in the *stable* ([GL88]) and the *wellfounded* ([GRS88]) semantics. The stable semantics in [GL88] was originally a two-valued semantics (truth values **t** and **f**), but it had the drawback that it was not universal (not all programs have a two-valued stable model). Therefore, this semantics was generalized to a three-valued stable semantics (see [PP90]): the third truth value is **u** (undefined or unknown). The stable semantics still has two more drawbacks:

- it is not unique (some programs have more than one stable model);
- finding a stable model is NP-hard (see [E89]).

The wellfounded semantics is unique and universal (and in general three-valued). If, moreover, the wellfounded model for a particular program is total (i.e. uses only the values **t** and **f**), then this model is also the unique two-valued stable model ([GRS88]). The analogy between the stable and the wellfounded semantics carries even further: the wellfounded model is the least three-valued stable model (see [P90], [W90]). Least here refers to the least amount of information, based on the truth value ordering $\mathbf{t} > \mathbf{u} < \mathbf{f}$.

Informally, the difference between the stable and the wellfounded semantics can be described as follows. In a stable model, there is in general not a specific reason for an atom to be false; in the

wellfounded model, however, an atom is only false if there can never be a reason to make it true. This is the way negation as failure works in the wellfounded model. In both models, an atom is only true if there is a reason for it.
We give an example, based on the general logic program

$$\sim a \rightarrow b$$
$$\sim b \rightarrow a$$

The stable models are $\{a \mapsto \mathbf{t}, b \mapsto \mathbf{f}\}$, $\{a \mapsto \mathbf{f}, b \mapsto \mathbf{t}\}$ and $\{a,b \mapsto \mathbf{u}\}$; the last model is also the wellfounded model.

The definition of the wellfounded model (see section 4) is in terms of a fixpoint of a monotonic operator which is defined using a union of subsets of the collection of atoms, called the *greatest unfounded set* (GUS). As a consequence, the naive implementation of the definition leads to an algorithm with exponential time complexity (given the size of the program), since all subsets have to be tested for inclusion in GUS. A slightly less naive approach - trying to determine membership of GUS pointwise - unfortunately does not work.
In [W90] Witteveen presents a quadratic-time algorithm for the wellfounded model. His work is based on an algorithm by Goodwin (see [G82]) in which the notion of wellfounded model is implicitly present (six years before the definition in [GRS88]), albeit restricted to the propositional case (as is Witteveen's algorithm). Witteveen proves that his algorithm yields the minimal three-valued stable model; T. Przymusinski proved in [P90] this model to be equal to his version of the wellfounded model, stating the equivalence to the original definition in [GRS88] without proof in [P89, Theorem 3.2]. [J91] contains our direct, but rather complicated proof that Witteveen's algorithm yields the wellfounded model. The present paper contains a more polished proof, based on an abstract formulation of the concepts involved.

2. Preliminaries

In this section we introduce our notation. The main orderd sets to be used here are presented in a table; explanation follows in the Remarks.

symbol	description	definition	variables	ordering
N	atoms	(some finite set of atoms)	a,b,c	trivial
T	truth values	{u,t,f,o}	x,y,z	u < t,f < o
I	interpretations	$\mathbf{N} \rightarrow \mathbf{T}$	I	inherited
J	justifications	$\{J : \mathbf{I} \rightarrow \mathbf{I} \mid J$ monotonic$\}$	J	inherited
B	clause bodies	$\wp(\mathbf{N} \times \sim\mathbf{N})$	$\alpha \,\&\, \sim\beta$	$\subseteq$
C	clauses	$\mathbf{B} \times \mathbf{N}$	$\alpha \,\&\, \sim\beta \rightarrow c$	inherited
CS	clause sets	$\wp(\mathbf{C})$	S	$\subseteq$
MCS	minimal clause sets	$\wp^{i}(\mathbf{C})$	S	(see below)

<u>2.1 Remarks</u>

1. $X \times Y$ and $X \to Y$ inherit their order from X and Y in the usual way:
for $z,z' \in X{\times}Y$, $z = \langle x,y\rangle$, $z' = \langle x',y'\rangle$ we have $z \le z'$ iff $(x \le x'$ and $y \le y')$;
for $f,g \in X \to Y$ we have $f \le g$ iff $\forall x{\in}X(fx \le gx)$.

2. For X an ondered set, $\wp^i(X)$ is the collection of al subsets of incomparable elements of X, i.e. $\wp^i(X) = \{Y \subseteq X \mid \forall yy'{\in}Y(y{\le}y' \to y{=}y')\}$. The ordering on $\wp^i(X)$ is defined by

$$A \le B \text{ iff } \forall x{\in}A\, \exists y{\in}B\, (x{\le}y).$$

3. We write $\sim\mathbf{N}$ for $\{\sim a \mid a \in \mathbf{N}\}$. p,q range over elements of $\mathbf{N}{\cup}{\sim}\mathbf{N}$.

4. The names of the truth values abbreviate **u**ndefined, **t**rue, **f**alse, **o**verdefined. In fact, taking only the consistent truth values **t**, **f**, **u** would suffice; **o** is only added for reasons of symmetry and elegance, e.g. to make $\oplus$ (sup) and $\otimes$ (inf) in 2.2 total.

5. For the purely mathematical part of the story only **N**, **T**, **I**, **J** would suffice. The other sets are introduced in order to formulate matters in the usual style of logic programming involving clauses. E.g. the elements of **C** are assumed to represent clauses directly, which is expressed by the choice of variables ranging over arbitrary elements (column 3).

2.2 Additional definitions

We have the following unary operations on **T**:

x	¬x	cx	|x|	+x	-x	Tx	Ux
u	u	t	u	u	u	t	u
t	f	u	t	t	u	t	u
f	t	u	t	u	f	t	u
o	o	u	t	t	f	t	u

As binary operations on **T** we have $\otimes$ (inf), $\oplus$ (sup) and & (logical and). $\otimes$ (inf) and $\oplus$ (sup) are defined as usual; & is defined as the unique symmetric extension of conjunction on $\{t,f\}$ over which $\otimes$ and $\oplus$ distribute, i.e.

$$(x{\oplus}y)\&z = (x\&z){\oplus}(y\&z)$$
$$(x{\otimes}y)\&z = (x\&z){\otimes}(y\&z).$$

$\otimes$, $\oplus$ and & are also used as prefix operators on sets of truth values, in the usual way, e.g. $\&\{x,y,z\} = x\&y\&z$.

All these operations are lifted to **I** as usual: e.g. $(\neg I)a = \neg(Ia)$ for $a{\in}\mathbf{N}$.
We put $\mathbf{T_c} = \{t,f,u\}$ (the consistent values) and $\mathbf{T_+} = \{t,u\}$ (the positive values).
We list some properties:

(1) $+\neg x = \neg\text{-}x$
(2) $x = +x \oplus -x$
(3) $c(x \oplus y) = cx \otimes cy$

(4) $|x| = ccx$
(5) $|x \oplus y| = |x| \oplus |y|$
(6) $x \otimes cx = |x| \otimes cx = u$
(7) $x \leq y \Leftrightarrow (+x \leq +y \text{ and } -x \leq -y)$

If, moreover, $x,y \in \mathbf{T}_+$, then

(8) $|x| = x$
(9) $x \leq y \Leftrightarrow x \otimes cy = u$
(10) $ccx = x$
(11) $x \oplus cx = t$

(1) - (11) can all be lifted in an obvious way to interpretations, reading I for x, I' for y, T for t and U for u. We also put

$$\mathbf{I_c} = \mathbf{N} \to \mathbf{T_c}$$
$$\mathbf{I_+} = \mathbf{N} \to \mathbf{T_+}$$
$$\mathbf{J_+} = \mathbf{J} \cap (\mathbf{I} \to \mathbf{I_+})$$

I is *consistent* with I' iff $I \oplus I' \in \mathbf{I_c}$; I is a consistent extension of I ($I \,_c\!\geq I'$) iff $I \in \mathbf{I_c}$ and $I \geq I'$. For $I, I' \in \mathbf{I_c}$ we have

(12) $I \oplus I' \in \mathbf{I_c} \Leftrightarrow I \otimes \neg I' = U.$

We shall use the fixpoint operator

$$\mathrm{fix} : \mathbf{J} \to \mathbf{I}$$
$$\mathrm{fix}(J) = \bigoplus\{J^n(\lambda a.u) \mid n \in \omega\} \text{ (the least fixpoint of } J)$$

with the properties

(13) $J(\mathrm{fix}(J)) = \mathrm{fix}(J);$
(14) if $J(I) \leq I$ then $\mathrm{fix}(J) \leq I.$

Here we used that **N** is finite, so the fixpoint of monotonic J is reached in finitely many steps.

3. Some isomorphisms

It is evident that **I** and **B** are isomorphic, by $\iota : \mathbf{B} \to \mathbf{I}$ and $\sigma : \mathbf{I} \to \mathbf{B}$ defined by

$$\iota(\alpha\&\sim\beta)(x) = \bigoplus(\{t \mid a \in \alpha\} \cup \{f \mid b \in \beta\})$$
$$\sigma(I) = \alpha_I \,\&\sim\beta_I \text{ where}$$
$$\alpha_I = \{x \mid I(x) \geq t\}$$
$$\beta_I = \{x \mid I(x) \geq f\}.$$

With this isomorphism in mind, we introduce the abbreviation

$$I \to c =_{def} \alpha_I \,\&{\sim}\beta_I \to c.$$

By appropriate restrictions of ι and σ, $\mathbf{I}_+$ and $\wp(\mathbf{N})$ are isomorphic. We let A range over elements of $\mathbf{I}_+$, sometimes considered as subsets of $\mathbf{N}$.

Now we shall show that $\mathbf{J}_+$ and **MCS** are isomorphic. Define $j : \mathbf{CS} \to \mathbf{J}_+$ by

$$\begin{aligned} j(S)(I)c &= t \text{ if } \exists I' \leq I\ (I' \to c \in S) \\ &= u \text{ otherwise.} \end{aligned}$$

It is clear that j is monotonic: if $S \leq S'$ and $j(S)(I)c = t$ then also $j(S')(I)c = t$; this holds for all $I \in \mathbf{I}$ and $c \in \mathbf{N}$, so $j(S) \leq j(S')$.
j has a right inverse, for e.g.

$$\text{if } S = \{I \to c \mid J(I)c = t\} \text{ then } j(S) = J$$

but this is in some sense not optimal: S is in general not minimal. But we do have

3.1. Lemma j is an isomorphism between $\mathbf{J}_+$ and **MCS**.
Proof: Define $cs : \mathbf{J}_+ \to \mathbf{MCS}$ by:

$$cs(J) = \{I \to c \mid J(I)c = t \ \& \ \forall I' < I\ J(I')c = u\}$$

We claim:
1. $cs(J) \in \mathbf{MCS}$
2. $J = j(cs(J))$ for all $J \in \mathbf{J}_+$
3. $S = cs(j(S))$ for all $S \in \mathbf{MCS}$
4. $J \leq J' \Leftrightarrow cs(J) \leq cs(J')$
This is proved as follows:
1. Directly from the definition of cs and **MCS**.
2. This follows from $\forall Ic\ (J(I)c = t \Leftrightarrow \exists I' \leq I\ (J(I')c = t \text{ and } \forall I'' < I'\ J(I'')c = u))$ and this is true. To see this, use (*): given I, there are only finitely many $I' \leq I$.
3. Using $S \in \mathbf{MCS}$, this comes down to $\forall Ic\ (I \to c \in S \Leftrightarrow (\exists I' \leq I\ (I' \to c \in S) \text{ and } \forall I'' < I\ (I'' \to c \notin S)))$, which follows with (*).
4. Analogously.

We introduce a validity relation. First we extend $\mathbf{I} : \mathbf{N} \to \mathbf{T}$ to $\mathbf{I} : \mathbf{B} \to \mathbf{T}$ by

$$I(\alpha\&{\sim}\beta) = \&\{Ip \mid p \in \alpha\&{\sim}\beta\}$$

Now we define

$$\begin{aligned} &\cdot \models \cdot : \mathbf{I_c} \times (\mathbf{B} \cup \mathbf{C} \cup \mathbf{CS} \cup \mathbf{I} \cup \mathbf{J}_+), \text{ defined by} \\ &I \models \alpha\&{\sim}\beta \qquad \text{iff } I(\alpha\&{\sim}\beta) \geq t \\ &I \models \alpha\&{\sim}\beta \to c \quad \text{iff } (I \models \alpha\&{\sim}\beta \Rightarrow Ic{=}t) \end{aligned}$$

$$
\begin{array}{ll}
I \models S & \text{iff } \forall(\alpha\&\sim\beta \to c)\in S\ (I \models \alpha\&\sim\beta \to c) \\
I \models I' & \text{iff } I \models \sigma(I') \\
I \models J & \text{iff } I \models cs(J)
\end{array}
$$

3.2. Lemma

i) If $I, I' \in \mathbf{I_c}$ then $I \models I'$ iff $I' \leq I$

ii) If $I \in \mathbf{I_c}$, $J \in \mathbf{J_+}$, then $I \models J$ iff $J(I) \leq I$

The proof of this lemma is easy and left to the reader.

A consequence of (ii) is: fix(J) is the least interpretation satisfying J, i.e. $\mathrm{fix}(J) \models J$, and if $I \models J$ then $\mathrm{fix}(J) \leq I$.

Using the isomorphisms above, we define the *immediate consequence* of interpretation I with respect to the clause set S by

$$\sigma(j(S)I)=\{c \mid I \models \alpha\&\sim\beta \text{ for some } \alpha\&\sim\beta\to c \text{ in } S\}.$$

4. Negation as failure via unfounded sets

As has been mentioned in the Introduction, the wellfounded semantics embodies negation as failure. Failure is captured by *unfounded* subsets of **N**, (modulo an interpretation I and a justification J).

First we define J*: it maps a positive interpretation $I \in \mathbf{I_+}$ to the collection of all immediate J-consequences of consistent extensions of I.

4.1. Definition

$.^* : \mathbf{J_+} \to \mathbf{J_+}$

$J^*(I) = \bigoplus\{J(I') \mid I' \ {}_c{\geq}\ I\}$.

It is clear that J^* is antimonotonic, for if $I \leq I'$, then $\{I'' \mid I'' \ {}_c{\geq}\ I\} \supseteq \{I'' \mid I'' \ {}_c{\geq}\ I'\}$, so (monotonicity of J and $\bigoplus$) $\bigoplus\{J(I'') \mid I'' \ {}_c{\geq}\ I\} \geq \bigoplus\{J(I'') \mid I'' \ {}_c{\geq}\ I'\}$.

Translating this to clauses we get:

$$\sigma(j(S)^*I) = \{c \mid \text{for some } \alpha\&\sim\beta\to c \text{ in } S \text{ we have } I(\alpha\&\sim\beta) \leq t\}.$$

4.2. Definition (unfoundedness)

unf : $\mathbf{I_+} \times \mathbf{I} \times \mathbf{J_+}$

$\mathrm{unf}(A,I,J) \Leftrightarrow \neg A \oplus J^*(I \oplus \neg A) \in \mathbf{I_c}$.

In words: $\neg A$ is consistent with all immediate J-consequences of conservative extensions of $I\oplus\neg A$. Formulated in terms of clauses:

$$\forall c\in A\ \forall \alpha\&\sim\beta\to c \in S\ (\ A\cap\alpha \neq \varnothing \text{ or } \neg a\oplus I \notin \mathbf{I_c} \text{ or } \alpha\cap\beta \neq \varnothing \text{ or } I(\alpha\&\sim\beta)\geq f\ ).$$

So, assuming

$\neg A \otimes I \in \mathbf{I_c}$ (OK if $A \subseteq \{c \mid Ic = u\}$) and
$\forall \alpha \& \sim\beta \rightarrow c \in S\ (\alpha \cap \beta \neq \varnothing)$,

we get the original definition (see [GRS88], [J91]):

$$\forall c \in A\, \forall \alpha \& \sim\beta \rightarrow c \in S\ (A \cap \alpha \neq \varnothing \text{ or } I(\alpha \& \sim\beta) \geq f).$$

Definition 4.2 only gives a way to check whether a set is unfounded, this check involves the justification (J) and takes linear time in the size of the program (see [GRS88}). There is however not a constructive definition of unfounded sets; i.e. no way is given how an unfounded set can be found.

4.3. Lemma
i) unf is monotonic in its second argument;
ii) in its first argument, unf is preserved under $\oplus$, i.e.
$\forall A \in X\ \mathrm{unf}(A,I,J) \Rightarrow \mathrm{unf}(\bigoplus X,I,J)\ \ (X \subseteq \mathbf{I_+})$.
Proof:
(i) Let $I \leq I'$ and assume $\mathrm{unf}(A,I,J)$, i.e. $A \otimes J^*(I \oplus \neg A) = U$; by $I \leq I'$ and the antimonotonicity of J^* we have $J^*(I \oplus \neg A) \geq J^*(I' \oplus \neg A)$, so $A \otimes J^*(I' \oplus \neg A) = U$, i.e. $\mathrm{unf}(A,I',J)$.
(ii)
$$\begin{aligned}
&\mathrm{unf}(\textstyle\bigoplus X,I,J)\\
\Leftrightarrow\quad & \qquad [\text{definition of unf}]\\
&\textstyle\bigoplus X \otimes J^*(I \oplus \sim\bigoplus X) = U\\
\Leftrightarrow\quad & \qquad [\text{property of } \textstyle\bigoplus]\\
&\forall A \in X\ \ A \otimes J^*(I \oplus \sim\textstyle\bigoplus X) = U\\
\Leftarrow\quad & \qquad [4.4\text{: } J^* \text{ is antimonotonic}]\\
&\forall A \in X\ \ A \otimes J^*(I \oplus \neg A) = U\\
\Leftrightarrow\quad & \qquad [\text{definition of unf}]\\
&\forall A \in X\ \ \mathrm{unf}(A,I,J).
\end{aligned}$$

4.4. Definition (greatest unfounded subset)
$\mathrm{Gus} : \mathbf{I} \times \mathbf{J_+} \rightarrow \mathbf{I_+}$
$\mathrm{Gus}(I,J) = \bigoplus\{A \in \mathbf{I_+} \mid \mathrm{unf}(A,I,J)\}$

Since definition 4.2 is not constructive, neither is this definition for Gus. The naive implementation of the definition leads to an algorithm with exponential time complexity (given the size of the program), since all subsets of **N** have to be tested for membership, as pointwise determination of membership of Gus is impossible.

4.5. Lemma
i) Gus is monotonic in its first argument;
ii) $\mathrm{unf}(\mathrm{Gus}(I,J),I,J)$, i.e. $\mathrm{Gus}(I,J) \otimes J^*(I \oplus \neg\mathrm{Gus}(I,J)) = U$;
iii) $J(I) \otimes \mathrm{Gus}(I,J) = U$
Proof: (i) follows directly from 4.7.(i).
(ii) Follows from 4.7.(ii) and the definition of Gus.

(iii) $J(I) \wedge Gus(I) = J(I) \otimes \bigoplus\{A \in \mathbf{I}_+ \mid A \otimes J^*(I \oplus \neg A) = U\}$ [definition of Gus and unf]
$= \bigoplus\{J(I) \otimes A \mid A \otimes J^*(I \oplus \neg A) = U\}$ [$\otimes$ distributes over $\bigoplus$]
$= U$ [$J(I) \leq J(I \oplus \neg A) \leq J^*(I \oplus \neg A)$]

Now we can define the wellfounded model wf(J) of J as the least fixpoint of the operator

$$J^{wf} : \mathbf{I} \to \mathbf{I}$$
$$J^{wf}(I) = J(I) \oplus \neg Gus(I,J)$$

Now $wf(J) = fix(J^{wf})$ is the smallest I satisfying

$$+I = J(I) \text{ and } -I = Gus(I,J);$$

this last property comes down to

$$unf(A,I,J) \Rightarrow \neg A \leq I,$$

i.e. if $A \neq \varnothing$ and $A \leq uI$ then there is a consistent extension I' of $I \oplus \neg A$ and an A with $Aa = J(I)a = t$, so I cannot be extended with $\neg A$.

5. A tractable approach

When trying to find an efficient algorithm for wf(J), one hits upon the problem of computing Gus(I,J). Direct implementation of the definition of Gus leads to exponential-time behavior: all subsets of **N** (coded as elements A of $\mathbf{I}_+$) have to be tested for unf(A,I,J). As unf is not antimonotonic in its first argument (i.e. we do not have unf(A,I,J) and $A' \leq A \Rightarrow unf(A',I,J)$), it is not possible to define Gus(I,J) pointwise by $\{a \mid unf(\iota\{a\},I,J)\}$, which would have resulted in a linear-time algorithm for Gus.

Witteveen presents in [W90] a linear-time algorithm for Gus(I,J) which is based on ideas by Goodwin ([G82]). It computes the complement of Gus as the least fixpoint of a monotonic operator which we call W (see 5.1). Using W, a quadratic-time algorithm is obtained in [W90] for the computation of wf(J) (see 5.5). In this section, we present our formulation of Witteveen's approach and derive the necessary properties.

5.1. Definition

$W : \mathbf{I} \times \mathbf{J}_+ \to \mathbf{J}_+$
$W(I,J)(A) = J^*(I \oplus \neg c(I \oplus A)) \otimes cI$

In terms of clauses:

$$cs(W(I,j(S))) = \{ \{a \in \alpha \mid Ia=u\} \to c \mid \alpha \& \sim\beta \to c \in S \text{ with } Ic=u \text{ and } I(\alpha \& \sim\beta) \leq t\}.$$

5.2. Lemma

i) $W(I,J): \mathbf{I} \to \mathbf{I}_+$ is monotonic;
ii) $W(I,J)(A) \leq A \Leftrightarrow unf(c(I \oplus A),I,J)$.

Proof: (i) Follows from the antimonotonicity of J^* and c.

(ii) $W(I,J)(A) \le A$

$\Leftrightarrow$ [definition of W]

$J^*(I \oplus \neg c(I \oplus A)) \otimes cI \le A$

$\Leftrightarrow$ [2.2. (9)]

$J^*(I \oplus \neg c(I \oplus A)) \otimes cI \otimes cA = U$

$\Leftrightarrow$ [2.2. (3)]

$J^*(I \oplus \neg c(I \oplus A)) \otimes c(I \oplus A) = U$

$\Leftrightarrow$ [definition of unf]

$unf(c(I \oplus A),I,J)$.

5.3. <u>Lemma</u> $fix(W(I,J)) = c(I \oplus Gus(I,J))$.

<u>Proof</u>: First we show $\ge$, then $\le$. We write F for $fix(W(I,J))$, G for $Gus(I,J)$.

$\ge$: $F \ge c(I \oplus G)$

$\Leftrightarrow$ [2.2. (3),(9)]

$G \ge c(I \oplus F)$

$\Leftarrow$ [definition of Gus]

$unf(c(I \oplus F),I,J)$

$\Leftrightarrow$ [lemma 5.2.(ii)]

$W(I,J)(F) \le F$

$\Leftrightarrow$ [property of fix]

true.

$\le$: $F \le c(I \oplus G)$

$\Leftarrow$ [property of fix]

$W(I,J)(c(I \oplus G)) \le c(I \oplus G)$

$\Leftrightarrow$ [lemma 5.2.(ii)]

$unf(G,I,J)$

$\Leftrightarrow$ [lemma 4.5.(ii)]

true.

So it seems that Gus(I,J) can be determined via its complement in **N**. We make this more precise:

5.4. <u>Corollary</u> $|I| \otimes Gus(I,J) = U \Rightarrow Gus(I,J) = c(I \oplus fix(W(I,J)))$

<u>Proof</u>: $c(I \oplus fix(W(I,J)))$

$=$ [lemma 5.3]

$c(I \oplus c(I \oplus Gus(I,J)))$

$=$ [2.2. (3),(5),(8); $Gus(I,J) \in \mathbf{I}_+$]

$cI \otimes (|I| \oplus Gus(I,J))$

$=$ [distribute $\otimes$ over $\oplus$]

$(cI \otimes |I|) \oplus (cI \otimes Gus(I,J))$

$=$ [2.2. (6); premiss $|I| \otimes Gus(I,J) = U$]

$Gus(I,J)$.

Now we obtain the interpretation gw(J) as the least fixpoint $fix(J^{gw})$, where $J^{gw} : \mathbf{I} \rightarrow \mathbf{I}$ is defined by

$$J^{gw}(I) = J(I) \oplus \neg c(+I \oplus fix(W(I,J))).$$

In order to show gw(J) = wf(J) we need

5.5. Lemma $I \leq J^{wf}(I) \Rightarrow J^{wf}(I) = J^{gw}(I) \leq J^{wf}(J^{wf}(I))$

Proof: Assume $I \leq J^{wf}(I)$. The inequality follows directly from the monotonicity of J^{wf}, so we look at the equality. We use the abbreviations $G = Gus(I,J)$, $F = fix(W(I,J))$. First we observe

$$\begin{array}{lll}
(*) & G \leq c{+}I & \\
 & \Leftrightarrow & \\
 & G \otimes {+}I = U & \\
 & \Leftarrow & \\
 & G \otimes J(I) = U \ \text{ and } \ {+}I \leq J(I) & \\
 & \Leftarrow & [\text{lemma 4.5.(iii); } {+}J^{wf}(I) = J(I)] \\
 & \text{true and } I \leq J^{wf}(I) &
\end{array}$$

and

$$\begin{array}{lll}
(**) & {-}I \leq \neg G & \\
 & \Leftarrow & [{-}J^{wf}(I) = {\sim}G \text{ and } 2.2.\ (7)\text{: } A \leq B \Leftrightarrow {+}A \leq {+}B \text{ and } {-}A \leq {-}B] \\
 & I \leq J^{wf}(I). &
\end{array}$$

Now

$$\begin{array}{lll}
J^{gw}(I) & = J(I) \oplus \neg c(+I \oplus F) & [\text{definition of } J^{gw}] \\
 & = J(I) \oplus \neg c(+I \oplus c(I \oplus G)) & [\text{lemma 5.3}] \\
 & = J(I) \oplus \neg(c{+}I \otimes (ccI \oplus G)) & [2.2.\ (3),(5),(8);\ G \in \mathbf{I}_+] \\
 & = J(I) \oplus \neg(c{+}I \otimes ccI) \oplus \neg(c{+}I \otimes G) & [\text{distribute } \otimes \text{ over } \oplus] \\
 & = J(I) \oplus \neg c(+I \oplus cI) \oplus \neg G & [\text{by } (*)] \\
 & = J(I) \oplus \neg G & [\text{by } (**) \text{ and } \neg c(+I \oplus cI) \leq -I \\
 & & \text{using truth tables}] \\
 & = J^{wf}(I) & [\text{definition of } J^{wf}]
\end{array}$$

5.6. Theorem $wf(J) = gw(J)$.

Proof: It suffices to show

$$\forall n\ (J^{wf})^n(U) = (J^{gw})^n(U)$$

and this follows with induction, using the previous lemma for the induction step (loaded with the condition $(J^{wf})^n(U) \leq (J^{wf})^{n+1}(U)$); the ground case is trivial.

6. Conclusion

In this paper we presented an abstract mathematical framework in which we proved two different definitions of the wellfounded model of propositional general logic programs to be equal, the second one leading to a quadratic-time algorithm. The main notions used are translated back in a style more common in the literature on logic programming.

At the moment, we apply this framework in the logical characterization of dependency-directed backtracking in reason maintenance systems, using the correspondence between propositional general logic programs and reason maintenance systems pointed out by Elkan in [E89].

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For the sake of completeness, we list some related definitions and results.

Definition
cl: $\mathbf{J} \to \mathbf{J}$
$cl(J)(I) = fix(\lambda I'.J(I')\oplus I)$.

Facts
i) $I \oplus J(cl(J)(I)) \leq cl(J)(I)$;
ii) if $I \oplus J(I') \leq I'$ then $cl(J)(I) \leq I'$.

Definition
$.^{\#} : \mathbf{J}_+ \to \mathbf{J}_+$
$J^{\#} = c(J^*)$.

Lemma $j(S)^{\#}(I) = N - j(S)^*(I) = \{c \mid \forall\ (\alpha\&\sim\beta\to c)\in S\ I(\alpha\&\sim\beta)\geq f\}$.

Lemma
i) J^* is antimonotonic, $J^{\#}$ is monotonic.
ii) $J^{\#\#} \geq J$.
iii) $J^{\#\#\#} = J^{\#}$.
Proof: i) If $I \leq I'$, then

$$\{I'' \mid I''\ {}_c\!\geq I\} \supseteq \{I'' \mid I''\ {}_c\!\geq I'\},$$

so (monotonicity of J and $\oplus$)

$$\bigoplus\{J(I') \mid I'\ {}_c\!\geq I\} \geq \bigoplus\{J(I') \mid I'\ {}_c\!\geq I\}$$

Since c is antimonotonic on $\{u,t\}$, we now obtain $J^{\#}(I) \leq J^{\#}(I')$.
ii) $J^{\#\#}(I) = c(\bigoplus\{c(\bigoplus\{J(I'') \mid I''\ {}_c\!\geq I'\}) \mid I'\ {}_c\!\geq I\})$
$= \bigotimes\{cc(\bigoplus\{J(I'') \mid I''\ {}_c\!\geq I'\}) \mid I'\ {}_c\!\geq I\}$
$= \bigotimes\{\bigoplus\{J(I'') \mid I''\ {}_c\!\geq I'\} \mid I'\ {}_c\!\geq I\}$
$\geq J(I)$
iii)

Lemma $j(S)^{\#\#}(I) = \{c \mid \forall I'_c\!\geq I\ \exists(\alpha\&\sim\beta\to c)\in S\ I'(\alpha\&\sim\beta)\leq t\}$
$= \{c \mid \forall f\in (Ker(I)\to\{u,t,f\})\ \exists(\alpha\&\sim\beta\to c)\in S\ (I\oplus f)(\alpha\&\sim\beta)\leq t\}$.

Definition
cons : $\mathbf{I} \times \mathbf{J}$
$cons(I,J) \Leftrightarrow I \oplus J^*(I) \in \mathbf{I}_c$.

Facts i) If $cons(I,J)$ and $I'\ {}_c\!\geq I$ then $I \oplus J(I') \in \mathbf{I}_c$;
ii) $cons(I,J) \Rightarrow cons(I,J^n)$;
iii) $cons(I,J) \Rightarrow cons(I,cl(J))$;
iv) $cons(I,J) \Rightarrow cl(J)(I) \in \mathbf{I}_c$.

Lemma Let $A \leq +uI$. Then

$$\mathrm{unf}(A,I,j(S)) \Leftrightarrow \forall c \in N\ \forall(\alpha\&\sim\beta\rightarrow c) \in S\ (A(c) = t \Rightarrow I(\alpha\&\sim\beta) = f \text{ or } \alpha \cap \alpha_A \neq \emptyset)$$

Proof: Assuming $A \leq +uI$, we have

$$\mathrm{unf}(A,I,j(S)) \Leftrightarrow A \otimes j(S)^*(I \oplus \neg A) = \emptyset$$
$$\Leftrightarrow A \otimes \bigoplus\{j(S)(I') \mid I' \,_c\!\geq I \oplus \neg A\} = \emptyset$$
etc.

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